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Performance of a Carnot refrigerator at maximum cooling power

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Abstract. The optimization procedure of finite-time thermodynamics is extended in order to calculate the coefficient of performance of ideal refrigerators at maximum cooling power. For this purpose, the cycle time has to be altered somewhat to allow for the occurrence of adiabatic arms in the cycle.

1. Introduction

In recent years great interest has arisen in the branch of finite-time thermodynamics (Andresen 1983, Andresen *et al* 1984, Callen 1985) which extends the usual reversible, infinitely slow thermodynamics to processes which are completed in a finite time, possibly in the presence of sources of irreversibility. Here the constraints arising from finite time, irreversibility, etc are conveniently modelled and a suitable functional e.g., power, efficiency or entropy production is optimized with respect to the parameters involved. Finite-time thermodynamics has been successfully applied to a large number of problems such as the analysis of heat engines of the endoreversible type (Curzon and Ahlborn 1975, Andresen *et al* 1977a, Andresen *et al* 1977b, Salamon *et al* 1980, 1982, Ondrechen *et al* 1981, Salamon and Nitzan 1981, Rubin and Andresen 1982, Mozurkewich and Berry 1982, Ondrechen *et al* 1983b, Hoffman *et al* 1985, Callen 1985, Leff 1987, Landsberg and Leff 1989), binary distillation processes (Andresen 1983, Andresen *et al* 1984), chemical reaction systems (Ondrechen *et al* 1980 and 1983a) separation processes (Brown *et al* 1986) and the upper bound on terrestrial wind energy (Gordon and Zarmi 1989).

The classic paper (Curzon and Ahlborn 1975) which initiated finite-time thermodynamics had discussed the important question of maximizing the power output of a finite-time, endoreversible Carnot engine by introducing temperature differences at the upper and lower isotherms. These authors wrote the cycle time as

$$\tau = \gamma(t_1 + t_3) \tag{1}$$

where t_1 and t_3 are the time-durations of the isothermal expansion and compression strokes, respectively, and γ is a constant. Using (1) they demonstrated how to maximize the power of a Carnot engine and also found that the efficiency at maximum power output is well approximated by the formula $1-(T_3/T_1)^{1/2}$. However, no attempt in the literature has been made so far to apply this philosophy to refrigerators where one would like to maximize the cooling power R (i.e., the rate of extraction of heat per cycle from the body to be cooled) so that the temperature of the body can achieve the desired lowest value as quickly as possible, and the aim of the present paper is to examine this task in detail.

In section 2 below we present the equations of cooling power optimization for a refrigerator based on the Curzon-Ahlborn (1975) technique and find that the equations do not admit realistic roots for the unknown temperature differences. We, therefore, suggest that the expression for τ(cf (1)) has to be altered somewhat by also explicitly calculating the times spent on the adiabatic arms of the cycle and show that consistent results are thereby obtained (section 3). For the sake of completeness we demonstrate in the appendix that the modified-time concept also works successfully for engines where the power output has to be maximized.

2. Curzon–Ahlborn technique applied to refrigerator

Following closely the notation of Curzon and Ahlborn (1975) we denote the source temperature by T_1 and the thermal conductance and the temperature difference at the upper isotherm by α and x, respectively. The corresponding quantities for the sink along with the lower isotherm are denoted by T_3, β and y, respectively. It should be emphasized that the refrigeration cycle (figure 1(a)) moves in a sense opposite to that of engines (figure 1(b)) namely it absorbs heat Q_3 at working temperature T_3-y, performs a work W on the same, and delivers heat $Q_1 = Q_3 + W$ at working temperature T_1+x. Then $\tau = \gamma[Q_1/\alpha x + Q_3/\beta y]$ and $Q_1/Q_3 = (T_1+x)/(T_3-y)$. The physically allowed domains of x and y are $0<x<\infty$ and $0<y<T_3$.

Now, we wish to maximize the cooling power R defined by

$$R = \frac{Q_3}{\tau} = \frac{\alpha\beta xy(T_3-y)}{\gamma[\beta T_1 y + \alpha T_3 x + xy(\beta-\alpha)]}. \tag{2}$$

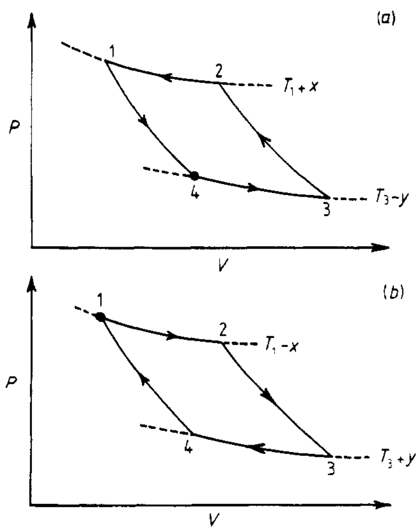

Figure 1. The Carnot cycle in the PV plane appropriate to (a) refrigerator and (b) engine. The starting point in either case is represented by the heavy dot.

Upon setting $\partial R/\partial y = 0$, one obtains

$$y = T_3\left[1 \pm \left(\frac{\beta}{\alpha}\frac{T_1 + x}{x}\right)^{1/2}\right]^{-1} \tag{3}$$

Since y must be less than T_3 only the positive sign before the radical has to be taken here. Also, equating $\partial R/\partial x$ to zero yields

$$\frac{\partial R}{\partial x} = \frac{T_1(T_3 - y)\alpha\beta^2 y^2}{\gamma[\beta T_1 y + \alpha T_3 x + xy(\beta - \alpha)]^2} = 0 \tag{4}$$

which is impossible because all the factors appearing on the left-hand side of (4) are finite. Of course, this conclusion is inevitable in view of the fact that R is of the form $Ax/(Bx + D)$. Hence the attempt to optimize the cooling power fails if we adopt the technique of Curzon and Ahlborn (1975).

It should be stressed that since physically acceptable roots for x and y do not exist in the case of refrigeration, there is no need to calculate the second-order derivatives in order to ascertain whether R becomes a genuine maximum. The physical reason for the failure of the above-mentioned optimization procedure can be ascribed to the fact that the total cycle time (cf (1)) has been calculated only in terms of contributions from the isothermal arms (cf figure 1) ignoring the adiabatic arms completely.

3. Modified time concept and refrigeration

3.1. Formulation

We suggest that the assumption behind (1), namely that the sum of the heat absorption and rejection times is a fixed fraction γ of the cycle time τ, is not strictly valid for a Carnot cycle because the adiabats and the isotherms have very different equations. To be precise, let the to-and-fro moving piston have a speed, i.e. the temporal rate of change of volume, u regarded as constant, neglecting the acceleration/deceleration at the ends of motion. This assumption is made for mathematical simplicity because a sinusoidal dependence of volume on the time would make the subsequent algebra much more complicated. Even physically the use of a piston with constant u can often improve the working of a machine as emphasized by Andresen *et al* (1984).

Referring to figure 1(a) the times t_3, t_2, t_1 and t_4 to go round the four branches of the refrigeration cycle are, by definition,

$$\begin{aligned} t_3 &= Q_3/\beta y = (V_3 - V_4)/u \qquad & t_2 &\equiv (V_3 - V_2)/u \\ t_1 &\equiv Q_1/\alpha x = (V_2 - V_1)/u \qquad & t_4 &\equiv (V_4 - V_1)/u. \end{aligned} \tag{5}$$

Now we make use of the well known relations characteristic of the Carnot cycle namely

$$\begin{aligned} Q_1/Q_3 &= (T_1 + x)/(T_3 - y) \equiv a \\ V_3/V_2 &= V_4/V_1 = [(T_1 + x)/(T_3 - y)]^C \end{aligned} \tag{6}$$

where $C \equiv C_V/R_g$ is the molar heat capacity at constant volume of the working substance in units of the gas constant R_g. Then, the above times on four arms can be summed to yield the total time as

$$\tau = 2Q_3[1 - (V_1/V_2)a^{-C}]/[\beta y(1 - V_1/V_2)] \tag{7}$$

and a relation between x and y in the form

$$y = [\alpha x a^{-C-1}]/\beta = [T_3 - T_1/a]/[1 + \beta a^C/\alpha]. \tag{8}$$

Obviously, in contrast to (1), our τ is not a simple multiple of $t_1 + t_3$. Furthermore, if x is regarded as the independent temperature difference then y gets fixed from (8).

Our desired expression for cooling power written conveniently in terms of 'a' reads

$$R=\frac{Q_3}{\tau}=\frac{\alpha(1-V_1/V_2)(T_3-T_1/a)}{2[a^C-V_1/V_2+\alpha/\beta-(\alpha V_1/\beta V_2)a^{-C}]}. \tag{9}$$

Keeping α, β, V_1/V_2, T_1, T_3 and C as fixed we differentiate R with respect to x, i.e. equivalently with respect to 'a' and equate the result to zero, getting the optimization condition

$$f(a)\equiv(T_1/a^2)[a^C-V_1/V_2+\alpha/\beta-(\alpha V_1/\beta V_2)a^{-C}] \\ -C(T_3-T_1/a)[a^{C-1}+(\alpha V_1/\beta V_2)a^{-C-1}]=0. \tag{10}$$

This algebraic equation does not possess an analytical solution for 'a' but can be readily solved numerically. The fact that we do get a genuine maximum for the cooling power R will become apparent in the next sub-section. Knowing a the coefficient of performance ω is obtained from

$$\omega=1/(a-1). \tag{11}$$

3.2. *Numerical results*

It should be emphasized that our theory applies to a four-arm Carnot cycle employing a non-latent-heat-type refrigerant which does not condense during the process. The best example is provided by air as the refrigerant (Sparks and Dulio 1959) which was used about a century ago for cold storage shipments of meats with the resulting coefficient of performance rarely exceeding 0.75. Recent advance in technology has been able to increase the value of the coefficient of performance to about 1.75 (Parker 1981) by using a cycle which is not exactly Carnot's. In order to illustrate our theory applied to four-arm air refrigeration we take (Vargaftik 1975)

$$T_1=316\text{ K}\qquad T_3=275\text{ K}\qquad C=2.558 \tag{12}$$

and study the following variations, after fixing the optimized value of a from (10) always. Since the compression ratio is of the order of 10-20 we take $V_2/V_1=16$ and plot (cf figure 2(a)) ω against α/β over a wide range of the heat conductance ratio. It is observed that ω increases gradually with decreasing α/β. In figure 2(b) we plot ω against V_2/V_1 for fixed $\alpha/\beta=2$ and find that ω slowly falls as V_2/V_1 becomes higher. The fact that we get a genuine maximum is obvious from figure 3 where the function R/α is plotted against 'a' for the typical choice $V_2/V_1=16$ and $\alpha/\beta=2$. The maximum corresponds to $a=1.79$, $x=121$ K and $y=31$ K and leads to

$$\omega=1.27\qquad \omega_C=6.7\qquad \omega_{obs}=0.75\text{-}1.75 \tag{13}$$

where $\omega_C=1/(T_1/T_3-1)$ is the standard Carnot performance factor and ω_{obs} is the value observed in practice (Sparks and Dulio 1959, Parker 1981). It is interesting to note that our ω agrees fairly well with ω_{obs}. However, for machines employing a non-latent-type refrigerant (namely air) published practical values of x and y are not available for direct comparison†. For the sake of completeness an application of our approach to engines is discussed in the appendix.

† The comparison between our theory and actual practice both for refrigerators and engines (see appendix) is not straightforward for several reasons. The cycle shapes in practice are not Carnot-like as the working fluid condenses on a portion of the cycle. In other words, the temperature parameters in modern machines are largely affected by the latent heat which is released/absorbed in contrast to our model which applies to non-latent-type gas. Moreover, various thermodynamic parameters of commercial machines are not based on the FTT optimization of the kind considered here. Nevertheless, to quote a typical practical example: the freon based refrigerator has T_1, T_3, x and y as 316, 275, 30 and 9 K, (Dossat 1961). Similarly, a steam engine plant has the corresponding values $T_1=457$, $T_3=300$, $x=132$ and $y=27$ K (Croft 1922).

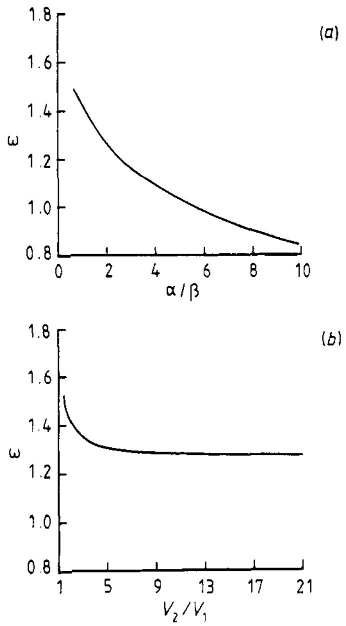

Figure 2. Plot of the coefficient of performance ω against (a) the heat conductance ratio α/β for fixed $V_2/V_1 = 16$ and (b) the expansion ratio V_2/V_1 at fixed $\alpha/\beta = 2$.

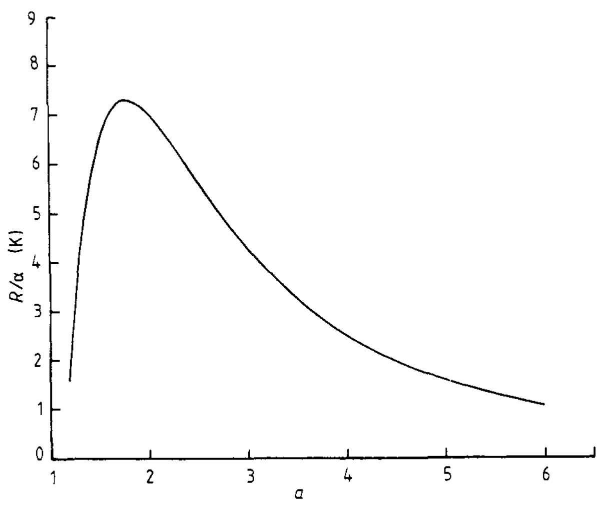

Figure 3. Plot of the ratio R/α for refrigerator against a keeping $V_2/V_1 = 16$ and $\alpha/\beta = 2$. The peak is located at $a = 1.79$ which corresponds to $\omega = 1.27$.

Before ending we wish to make some pertinent remarks. One may wonder why the specific choice $\alpha/\beta = 2$ was made in connection with refrigerator (cf figure 3) and engine (cf figure 5) though the results of ω or η are insensitive to it. Since the heat transferred along the upper isotherm is always larger than that along lower isotherm the corresponding conductances should also bear this pattern. Next, with a slight modification of the concept of cycle time we have extended successfully the classic work of Curzon and Ahlborn (1975) to refrigeration also without disturbing the requirement that the temperature difference x at the upper isotherm is unbounded. Finally, our plea is that the parameters predicted by finite-time thermodynamics should be adopted by commercial manufacturers of engines and refrigerators so as to improve their performance characteristics. It is hoped that the present work will serve as a base for more elaborate calculations involving realistic cycles and general working substances.

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Appendix. Modified time concept for engines

In the engine heat Q_1 is absorbed at working temperature $T_1 - x$, a part W of it gets converted into work, and the remainder $Q_3 = Q_1 - W$ is delivered at working temperature $T_3 + y$ as shown in figure 1(b). The calculation of the cycle time proceeds in a manner exactly analogous to (7) except that the variable 'a' is now defined as

$$a = (T_1 - x)/(T_3 + y). \tag{A1}$$

Furthermore, the output power P reads

$$P = \frac{\alpha}{2}\frac{(1 - V_1/V_2)(a-1)(T_1/a - T_3)}{[a^C - V_1/V_2 + \alpha/\beta - (\alpha V_1/\beta V_2)a^{-C}]}. \tag{A2}$$

The power maximization condition namely, $\partial P/\partial a = 0$ yields

$$\begin{aligned} g(a) \equiv {} & [T_3 - T_1/a^2][a^C - V_1/V_2 + \alpha/\beta - (\alpha V_1/\beta V_2)a^{-C}] \\ & - C[T_3(a-1) + T_1(1/a - 1)][a^{C-1} + (\alpha V_1/\beta V_2)a^{-C-1}] \\ = {} & 0 \end{aligned} \tag{A3}$$

the root of which that is smaller than T_1/T_3 is chosen for this purpose. The efficiency is, of course, given by

$$\eta = 1 - 1/a. \tag{A4}$$

For numerical illustration we choose the example of West Turrock (UK) Coal Fired Steam Plant (Spalding and Cole 1966) considered also by Curzon and Ahlborn (1975) for which data on specific heat (Vargaftik 1975) of the working substance are available. Taking

$$T_1 = 838\ \text{K} \qquad T_3 = 298\ \text{K} \qquad C = 3.767 \tag{A5}$$

we depict in figure 4(a) the dependence of η on α/β for fixed $V_2/V_1 = 16$. Figure 4(b) displays the variation of η with V_2/V_1 keeping $\alpha/\beta = 2$. Once again, the dependences of η on the plotted parameters are very weak. The fact that the output power P becomes a genuine maximum is evident from the plot of P/α against 'a' in figure 5 corresponding

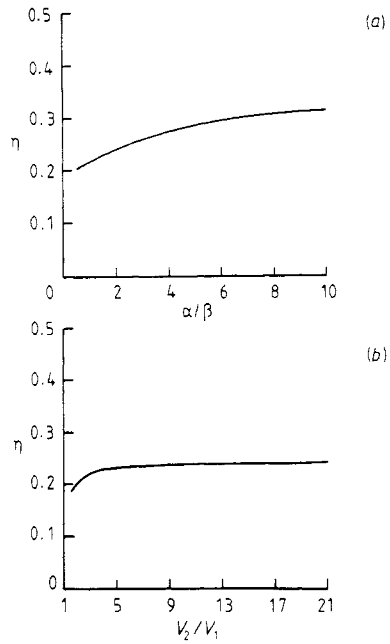

Figure 4. The efficiency η of an engine plotted against (*a*) α/β for fixed $V_2/V_1=16$ and (*b*) V_2/V_1 at fixed $\alpha/\beta=2$.

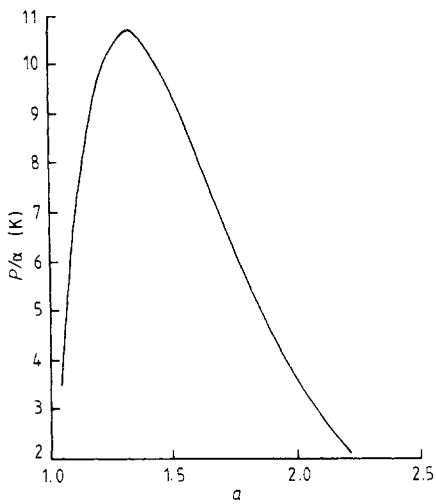

Figure 5. Plot of the ratio P/α against a keeping $V_2/V_1=16$ and $\alpha/\beta=2$ for an engine. The peak is located at $a=1.32$ which corresponds to $\eta=24\%$.

to the realistic choice $V_2/V_1=16$ and $\alpha/\beta=2$. The peak value corresponds to $a=1.32$, $x=261$ K, $y=139$ K and leads to

$$\eta=24\% \qquad \eta_{\text{obs}}=36\% \qquad \eta_{\text{C}}=64\% \qquad \eta_{\text{CA}}=36\% \tag{A6}$$

where the subscripts obs, C and CA refer to the observed, Carnot, and Curzon-Ahlborn values, respectively. Once again η is fairly close to η_{obs} but the comparison between the theoretical and practical values of x and y is not straightforward as mentioned in the footnote of subsection 3.2.

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